Scaling Solutions in Robertson-Walker Spacetimes

R. J. van den Hoogen
Department of Mathematics, Statistics and Computer Science,
Saint Francis Xavier University, Antigonish, Nova Scotia, B2G 2W5

A. A. Coley
Department of Mathematics, Statistics and Computing Science
Dalhousie University, Halifax, Nova Scotia B3H 3J5

D. Wands
School of Computer Science and Mathematics
University of Portsmouth, Portsmouth, PO1 2EG, U.K.

We investigate the stability of cosmological scaling solutions describing a barotropic fluid with $p = (\gamma - 1)\rho$ and a non-interacting scalar field ϕ with an exponential potential $V(\phi) = V_0 \mathrm{e}^{-\kappa\phi}$. We study homogeneous and isotropic spacetimes with non-zero spatial curvature and find three possible asymptotic future attractors in an ever-expanding universe. One is the zero-curvature power-law inflation solution where $\Omega_\phi = 1$ ($\gamma < 2/3, \kappa^2 < 3\gamma$ and $\gamma > 2/3, \kappa^2 < 2$). Another is the zero-curvature scaling solution, first identified by Wetterich, where the energy density of the scalar field is proportional to that of matter with $\Omega_\phi = 3\gamma/\kappa^2$ ($\gamma < 2/3, \kappa^2 > 3\gamma$). We find that this matter scaling solution is unstable to curvature perturbations for $\gamma > 2/3$. The third possible future asymptotic attractor is a solution with negative spatial curvature where the scalar field energy density remains proportional to the curvature with $\Omega_\phi = 2/\kappa^2$ ($\gamma > 2/3, \kappa^2 > 2$). We find that solutions with $\Omega_\phi = 0$ are never late-time attractors.

PACS numbers(s): 04.20.Jb, 98.80.Hw gr-qc/9901014

I. INTRODUCTION

There have been a number of studies of spatially homogeneous scalar field cosmological models with an exponential potential, with particular emphasis on the possible existence of inflation in such models if the potential is sufficiently flat [1–5]. Although modern supergravity theories generally predict exponential potentials that are too steep to drive inflation, these models still have important cosmological consequences. For example, in models with barotropic matter, such as dust or radiation, there exist spatially homogeneous and isotropic 'scaling solutions' in which the scalar field energy density tracks that of the perfect fluid [6–9], so that a significant fraction of the energy density of the Universe at late times may be contained in the homogeneous scalar field whose dynamical effects mimic the barotropic matter. The tightest constraint on these cosmological models comes from primordial nucleosynthesis bounds on any such relic density during the radiation dominated era [6]. More recently attention has been directed to the possible effect of such a scalar field on the growth of large-scale structure in the universe [9–12].

A phase-plane analysis of the spatially homogeneous and isotropic *zero curvature* models [8] has shown that these scaling solutions are the unique late-time attractors whenever they exist. The stability of these scaling solutions in more general spatially homogeneous cosmological models was studied in [13]. In this article we shall study scaling solutions in the general class of spatially homogeneous and isotropic models with non-zero curvature.

The governing equations for a self-interacting scalar field with an exponential potential energy density

$$V = V_0 e^{-\kappa\phi}, \tag{1.1}$$

where V_0 and κ are positive constants, evolving in a Robertson-Walker spacetime containing a separately conserved perfect fluid, are given by

$$\dot{H} = -\frac{1}{2}(\gamma\rho_\gamma + \dot{\phi}^2) - K, \tag{1.2}$$

$$\dot{\rho}_\gamma = -3\gamma H \rho_\gamma, \tag{1.3}$$

$$\ddot{\phi} = -3H\dot{\phi} + \kappa V, \tag{1.4}$$

subject to the Friedmann constraint

$$H^2 = \frac{1}{3}(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V) + K, \tag{1.5}$$

where $K = -kR^{-2}$ and k is a constant that can be scaled to 0, ± 1, $H \equiv \dot{R}/R$ is the Hubble parameter, an overdot denotes ordinary differentiation with respect to time t, and units have been chosen so that $8\pi G = 1$. In the above we have assumed that the perfect fluid satisfies the barotropic equation of state

$$p_\gamma = (\gamma - 1)\rho_\gamma, \tag{1.6}$$

where γ is a constant which satisfies $0 < \gamma < 2$. We also note that the total energy density of the scalar field is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V. \tag{1.7}$$

Defining

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H} \quad , \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}H} \quad , \quad \Omega \equiv \frac{\rho_\gamma}{3H^2}, \tag{1.8}$$

and the new logarithmic time variable τ by

$$\frac{d\tau}{dt} \equiv H, \tag{1.9}$$

equations (1.2–1.4) can be written as the three-dimensional autonomous system:

$$x' = -3x + \sqrt{\frac{3}{2}}\kappa y^2 + \frac{3}{2}x\left[\left(\gamma - \frac{2}{3}\right)\Omega + \frac{2}{3}(1 + 2x^2 - y^2)\right], \tag{1.10}$$

$$y' = \frac{3}{2}y\left[-\sqrt{\frac{2}{3}}\kappa x + \left(\gamma - \frac{2}{3}\right)\Omega + \frac{2}{3}(1 + 2x^2 - y^2)\right], \tag{1.11}$$

$$\Omega' = 3\Omega\left[\left(\gamma - \frac{2}{3}\right)(\Omega - 1) + \frac{2}{3}(2x^2 - y^2)\right], \tag{1.12}$$

where a prime denotes differentiation with respect to τ, and equation (1.5) becomes

$$1 - \Omega - x^2 - y^2 = KH^{-2}, \tag{1.13}$$

where

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = x^2 + y^2. \tag{1.14}$$

.

II. QUALITATIVE ANALYSIS

A. Invariant Sets and Monotone Functions

The physical region of the state-space is constrained by the requirement that the energy density be non-negative; i.e., $\Omega \geq 0$. Furthermore, from equation (1.13) we find that in the variables used the state-space is bounded for $k = 0$ and $k = -1$ but not for $k = +1$.

Geometrically the zero-curvature models ($k = 0$) are represented by the paraboloid

$$\Omega + x^2 + y^2 = 1, \tag{2.1}$$

in the (Ω, x, y) state-space. It is therefore possible to divide the complete state-space up into a number of invariant sets. (See Table I.)

The existence of a monotone function in any invariant set rules out the existence of periodic orbits, recurrent orbits, and equilibrium points in that set and serves to determine the asymptotic behaviour in that set [14,15]. In the three-dimensional sets $\mathcal{A}$ and $\mathcal{C}$ we define the function

$$Z = \frac{\Omega^2}{(\Omega + x^2 + y^2 - 1)^2}. \tag{2.2}$$

From Eqns. (1.10-1.12) we find that

$$Z' = 2(2 - 3\gamma)Z. \tag{2.3}$$

For $\gamma > 2/3$ we have that Z is a monotone decreasing function along orbits in sets $\mathcal{A}$ and $\mathcal{C}$, which therefore implies that $\Omega \to 0$ along these orbits. For $\gamma < 2/3$ we have that Z is a monotone increasing function along orbits in sets $\mathcal{A}$ and $\mathcal{C}$ which also implies that $K/6H^2 \to 0$ along these orbits. (The case $\gamma = 2/3$ will be dealt with separately in Sec. II C.) Hence we observe that in the non-zero-curvature models either the curvature approaches zero or the energy density of the fluid approaches zero as time evolves forward. That is, the asymptotic behaviour of these models can be completely determined by analyzing the zero-curvature models (see [8]), and the scalar field models with no fluid matter (see [2–5]).

B. Local Stability of the Equilibrium Points

Due to the existence of the monotonic function Eq. (2.2), all we need to do is locally analyze the equilibrium points of the system (1.10–1.12). There are seven equilibrium points, labelled P1–P7. The equilibrium points and their local stability are listed in Table II.

The exact solution corresponding to the equilibrium point **P1** is the standard **Milne model**. Equilibrium points **P2** and **P3** correspond to a massless scalar field models which are essentially equivalent to **stiff perfect fluid Friedmann-Robertson-Walker models**. The remaining equilibrium points and their corresponding exact solutions are discussed later in the text.

C. The Bifurcation Values

Bifurcations of the the system (1.10–1.12) occur for various values of the equation of state parameter for the perfect fluid, γ, and the scalar field potential parameter, κ. The bifurcation values are $\gamma = 2/3$, $\kappa^2 = 2$, $\kappa^2 = 6$, and $\kappa^2 = 3\gamma$. Each of these bifurcation values will be discussed in turn.

1. $\gamma = 2/3$

If $\gamma = 2/3$ then the equilibrium points of the system (1.10–1.12) are points **P2**, **P3**, **P4** as well as the non-isolated lines of equilibria given by

$$\mathbf{L1} = (\Omega = \Omega_0, x = 0, y = 0)$$

and

$$\mathbf{L2} = (\Omega = \Omega_s, x = \frac{\sqrt{2}}{\sqrt{3}\kappa}, y = \frac{2}{\sqrt{3}\kappa})$$

where $0 \leq \Omega_0, \Omega_s < \infty$. These two lines of equilibria are the degeneracies of equilibrium points **P1**, **P5**, **P6** and **P7**. Fortunately the analysis of the dynamics of the system are simplified through the observation that the function defined by Eq. (2.2) is constant when $\gamma = 2/3$. This implies that the dynamics of the system (1.10–1.12) are restricted to level surfaces of the function $Z = Z(x, y, \Omega)$. Essentially this implies that dynamics of the three-dimensional system (1.10–1.12) can be thought of as a one-parameter family of two-dimensional surfaces (paraboloids in this case), where the dynamics on each surface are identical and equivalent to the dynamics on any other surface, (including the surfaces represented by $Z = 0$ and $1/Z = 0$). The future asymptotic attractor for all models is the power-law inflationary model represented by equilibrium point **P4** if $\kappa^2 < 2$. If $\kappa^2 > 2$ then the matter scaling solutions represented by the line of equilibria **L2** are the future asymptotic attractors for all models.

2. $\kappa^2 = 2$

If $\kappa^2 = 2$ then points **P4** and **P5** coalesce. As $\kappa^2 \to 2^-$, **P5** $\to$**P4**, and as κ^2 increases past 2, the stability of one of the eigendirections changes. In essence **P4** and **P5** undergo a transcritical bifurcation [16]. The point **P4** is a saddle node when $\kappa^2 = 2$. If $\kappa^2 = 2$ and $\gamma > 2/3$ then both the negative and zero curvature models tend to the equilibrium point **P4**. For the positively curved models, this same point acts like a saddle with a one-dimensional unstable manifold when $\kappa^2 = 2$ and $\gamma > 2/3$. If $\kappa^2 = 2$ and $\gamma < 2/3$ then all models are attracted to the matter scaling solution represented by **P7**.

3. $\kappa^2 = 6$

If $\kappa^2 = 6$, then equilibrium points **P4** and **P2** coalesce. However, the future asymptotic behaviour of the system can be determined directly from Table II.

4. $\kappa^2 = 3\gamma$

If $\kappa^2 = 3\gamma$ then **P7** and **P4** coalesce. If $\gamma < 2/3$ then this point attracts those orbits in the physical state space, i.e., it is the future attractor. If $\gamma > 2/3$ then there exists a one-dimensional unstable manifold. If $\kappa^2 = 3\gamma$ and $\gamma > 2/3$ then the negatively curved models are attracted to the point **P5**, the zero curvature models are attracted to the point **P4**.

D. Zero-Curvature Models

The zero-curvature models are contained in the two-dimensional invariant set $\mathcal{B} \cup \mathcal{E}$. A qualitative analysis of this plane-autonomous system is given in [8]. The well-known **power-law inflationary solution** for $\kappa^2 < 2$ [2–5] corresponds to the equilibrium point **P4**, which is shown to be stable (i.e., attracting in the two-dimensional invariant set $\mathcal{B} \cup \mathcal{E}$) for $\kappa^2 < 3\gamma$ in the presence of a barotropic fluid.

In addition, for $0 < \gamma < 2$ there exists a **matter scaling solution** corresponding to the equilibrium point **P7**, whenever $\kappa^2 > 3\gamma$ [6]. The equilibrium point is stable in the two-dimensional invariant set $\mathcal{B} \cup \mathcal{E}$, (a spiral for $\kappa^2 > 24\gamma^2/(9\gamma - 2)$, otherwise a node) so that the corresponding cosmological solution is a late-time attractor in the class of zero-curvature models in which neither the scalar-field nor the perfect fluid dominates the evolution and we have

$$\Omega_\phi = \frac{3\gamma}{\kappa^2}\ . \tag{2.4}$$

The effective equation of state for the scalar field is given by

$$\gamma_\phi \equiv \frac{(\rho_\phi + p_\phi)}{\rho_\phi} = \frac{2x^2}{x^2 + y^2}$$

which is the same as the equation of state parameter for the perfect fluid at this equilibrium point; i.e., $\gamma_\phi = \gamma$. The solution is referred to as a matter scaling solution since the energy density of the scalar field remains proportional to that of the barotropic perfect fluid according to $\rho_\gamma/\rho_\phi = (\kappa^2 - 3\gamma)/3\gamma$ [6]. Since this matter scaling solution corresponds to an equilibrium point of the system (1.10–1.12), we note that it is a self-similar cosmological model [14].

E. Non-Zero Curvature Models

1. $\gamma > 2/3$

If $\kappa^2 < 2$, then the future asymptotic state for the negative-curvature models and a subset of the positive-curvature models is the standard zero-curvature power-law inflationary solution, represented by equilibrium point **P4**, where $\Omega_\phi = 1$. Previous analysis has shown that this power-law inflationary solution is a global attractor in spatially

homogeneous models in the absence of a perfect fluid (except for a subclass of Bianchi type IX models which recollapse) [4]. However, more recent analysis has shown that this power-law inflationary solution is also a global attractor in spatially homogeneous Bianchi class B models (as classified by Ellis and MacCallum [17]) in the presence of a perfect fluid [18] with the same restrictions on γ and κ as above.

If $\kappa^2 > 2$, then the negative-curvature models are asymptotic towards the **curvature scaling solution**, represented by equilibrium point **P5**, where

$$\Omega_\phi = \frac{2}{\kappa^2}\ . \tag{2.5}$$

Part of the motivation here is to study the stability of these scaling solutions. It is known that the equilibrium point **P5** is the future asymptote for the Bianchi type V and Bianchi type VII$_h$ models (see [13,18]).

2. $\gamma < 2/3$

If $\kappa^2 < 3\gamma$, then power-law inflation, represented by equilibrium point **P4**, is again the future asymptotic state for the negative-curvature models and a subset of the positive-curvature models. But for $\kappa^2 > 3\gamma$ the matter scaling solution, represented by equilibrium point **P7**, takes over as the future asymptotic attractor.

The linearization of system (1.10–1.12) about the equilibrium point **P7** yields two negative real eigenvalues and the eigenvalue $(3\gamma - 2)$. Hence the matter scaling solution is only stable for $\gamma < \frac{2}{3}$. For $\gamma > \frac{2}{3}$ the equilibrium point **P7** is a saddle with a two-dimensional stable manifold (lying in the set $\mathcal{B} \cup \mathcal{E}$ representing the zero-curvature models) and a one-dimensional unstable manifold.

III. COSMOLOGICAL IMPLICATIONS

In order to solve the flatness problem of the standard model, a period of accelerated expansion where the the spatial curvature is driven to zero ($K/H^2 \to 0$) is desirable [19]. Power-law models of inflation driven by scalar fields with exponential potentials with $\kappa^2 < 2$ provide an interesting model of inflation where exact analytic solutions are possible, not only for the background homogeneous fields [1], but also for inhomogeneous linear perturbations [20]. We have shown that the power-law inflation solution, represented by equilibrium point **P4**, with $\Omega_\phi = 1$ and zero curvature, is a stable late time attractor solution in models with spatial curvature and non-zero density of barotropic fluid for $\gamma > 2/3$ and $\kappa^2 < 2$ and for $\gamma < 2/3$ and $\kappa^2 < 3\gamma$.

Power-law inflation can also be driven by a barotropic fluid with $\gamma < 2/3$. However, we have shown that this fluid dominated solution, corresponding to equilibrium point **P6**, is never stable for $\gamma > 0$ in the presence of a scalar field with an exponential potential. Instead, for $\gamma < 2/3$ and $\kappa^2 > 3\gamma$ we have shown that the matter scaling solution, corresponding to equilibrium point **P7** with zero spatial curvature, is the late time attractor for all models with zero or negative curvature, and for a subset of models with positive curvature.

This leaves a possible relic density problem [8]. Although the spatial curvature may become negligible, the energy density in a scalar field with an exponential potential cannot be diluted away by inflation driven by barotropic matter unless $\gamma \to 0$. In typical slow-roll inflation models $\gamma \sim 0.1$ [21], and thus the exponential potential can have a non-negligible density during and after inflation. During the subsequent radiation dominated era ($\gamma = 4/3$) the scalar field rapidly approaches the matter scaling solution where $\Omega_\phi = 4/\kappa^2$. Models of primordial nucleosynthesis require $\Omega_\phi < 0.13$ to 0.2 [9]. Taking the more conservative upper limit we can conclude that the existence of scalar fields with exponential potentials is incompatible with standard models of nucleosynthesis for $\kappa^2 < 20$, unless the scalar field has for some reason not reached its scaling solution. Conventional inflation models with $0 < \gamma < 2/3$ cannot prevent Ω_ϕ reaching its equilibrium value soon after inflation.

Observational evidence suggests we live in a Universe with a present day energy-density Ω_0 bounded between 0.1 and 0.3 [22]. An intriguing possibility is that the dominant dark matter in the Universe could be in the form of a scaling scalar field with an exponential potential, which would be compatible with a low density, but spatially flat universe for $\kappa^2 \sim 4$. However, this would only be compatible with the nucleosynthesis bounds quoted above if the scalar field is far from its scaling solution at the time of nucleosynthesis [9]. Such a model is also likely to be in conflict with conventional models of structure formation as an inhomogeneous scalar field behaves differently from the barotropic matter and scalar field gradients can exert large pressures that resist gravitational collapse on small scales [12,10].

We have shown that the matter scaling solution is always unstable to curvature perturbations in the presence of ordinary matter ($\gamma \geq 1$); i.e., the matter scaling solution is no longer a late-time attractor in this case. However, it

does still correspond to an equilibrium point in the governing autonomous system of ordinary differential equations and hence there are cosmological models that can spend an arbitrarily long time ‘close’ to this solution. Indeed the curvature of the Universe is presently constrained to be small by cosmological observations, so it is possible that the matter scaling solution could be important in the description of our actual Universe. Negative curvature spaces are interesting since they can support a variety of possible topologies, leading to, for example, hyperbolic null geodesic motions which can produce interesting patterns in the cosmic microwave background radiation [23]. At late times, all models with $\kappa^2 > 2$, $\gamma > 2/3$ and negative spatial curvature approach the curvature scaling solution, corresponding to equilibrium point **P5**.

Classical cosmological tests such as number counts, or the luminosity of standard candles, should in principle be able to distinguish between the different possible low-density models. The deceleration parameter at any point in the phase-space is given by

$$q_0 \equiv -\frac{\ddot{R}R}{\dot{R}^2} = \left(\frac{3\gamma}{2} - 1\right)\Omega + 2x^2 - y^2 \ . \tag{3.1}$$

This reduces to the standard result $q_0 = 3\gamma/2 - 1$ for the matter scaling solution, or $q_0 = 0$ for the curvature scaling solution. The dynamics of the cosmological scale factor in these these solutions is indistinguishable from the matter dominated or curvature dominated solutions. However the luminosity distance as a function of redshift depends not only upon the evolution of the scale factor, but also upon the spatial geometry, and hence the curvature scaling solution can in principle be distinguished from the curvature dominated Milne model.

Recent results from high redshift supernovae searches suggest that the deceleration parameter is negative at present; i.e., the Universe is in fact accelerating [24]. Although it is possible to obtain $q_0 < 0$ when $y^2 > 2x^2$, we have found from numerical experimentation that this is unlikely during a transition from matter scaling to curvature scaling regime when $\kappa^2 > 2$. Instead it would only seem to be compatible with either barotropic matter with $\gamma < 2/3$ or a scalar field dominated solution, corresponding to equilibrium point **P4**, with $\kappa^2 < 2$.

IV. CONCLUSIONS

A complete qualitative analysis of the dynamical system describing the evolution of spatially homogeneous and isotropic models containing a non-interacting perfect fluid and a scalar field with an exponential potential has yielded the following results:

1. The past asymptotic state for all expanding models is the massless scalar field solution.
2. The future asymptotic state depends upon the values of the equation of state parameter for the fluid, γ, and the steepness of the potential κ. The future asymptotic state is either
   (a) the standard power-law inflationary solution (if $\gamma < 2/3$, $\kappa^2 < 3\gamma$ or $\gamma > 2/3$, $\kappa^2 < 2$),
   (b) a matter scaling solution (if $\gamma < 2/3$, $\kappa^2 > 3\gamma$), or
   (c) in the case of the negative-curvature models, a curvature scaling model with no barotropic matter ($\gamma > 2/3$, $\kappa^2 > 2$).
3. In non-zero curvature models with ordinary matter, (i.e., $\gamma \geq 1$), the matter scaling solution is never stable.

Current cosmological observations seem to indicate $\Omega < 1$ in ordinary matter which could be due to the presence of (negative) spatial curvature, and or some other form of energy density such as a scalar field with exponential potential. It is known that the matter scaling solutions are late-time attractors (i.e., stable) in the subclass of zero-curvature isotropic models [8]. It is also known that the matter scaling solutions are stable (to shear and curvature perturbations) in generic anisotropic Bianchi models when $\gamma < 2/3$ [13]. However, when $\gamma > 2/3$, and particularly for ordinary matter with $\gamma \geq 1$, the matter scaling solutions are unstable; essentially they are unstable to curvature perturbations, although they are stable to shear perturbations [13]. Even though it is unstable, this matter scaling solution may still play an important role in describing the intermediate (or transient) physics between the past and future asymptotic attractors.

The negative curvature scaling solutions may also be of importance since they are self-similar cosmological models corresponding to an equilibrium point of the dynamical system (1.10–1.12) where the scalar field still has a non-vanishing contribution to the energy density. Cosmological tests such as the magnitude-redshift relation for standard candles or detailed observations of the microwave background should in principle be able to determine whether these solutions could describe the present state of the universe.

ACKNOWLEDGMENTS

RvdH and AAC were supported by the Natural Sciences and Engineering Research Council of Canada. RvdH was also supported by a grant through the St.F.X. University Council on Research. RvdH would like to thank Colleen Meagher and Bill MacMillan for checking the analysis and doing some numerical calculations.

[1] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).
[2] J.J. Halliwell, Phys. Lett. B**185**, 341 (1987);
[3] A.B. Burd and J.D. Barrow, Nucl. Phys. B**308**, 929 (1988);
[4] Y. Kitada and K. Maeda, Class. Quantum Grav. **10**, 703, (1993);
[5] A.A. Coley, J. Ibañez and R.J. van den Hoogen, J. Math. Phys. **38**, 5256 (1997).
[6] C. Wetterich, Nucl. Phys. B**302**, 668 (1988).
[7] D. Wands, E.J. Copeland and A.R. Liddle, Ann. N.Y. Acad. Sci. **688**, 647 (1993).
[8] E.J. Copeland, A.R. Liddle and D. Wands, Phys. Rev. D. **57**, 4686 (1998).
[9] P.G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998).
[10] P.G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997).
[11] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).
[12] P. T. P. Viana and A. R. Liddle, Phys. Rev. D **57**, 674 (1998).
[13] A.P. Billyard, A.A. Coley and R.J. van den Hoogen, to appear in Phys. Rev. D., (1998).
[14] J. Wainwright and G.F.R. Ellis, *Dynamical Systems in Cosmology*,(Cambridge University Press, 1997).
[15] J. Hale, *Ordinary Differential Equations*, (J. Wiley and Sons, 1969).
[16] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, (Springer, 1990).
[17] G.F.R. Ellis, M.A.H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
[18] A.P. Billyard, A.A. Coley, R.J. van den Hoogen, J. Ibañez, and I. Olasagasti, Preprint (1998).
[19] E. W. Kolb and M. S. Turner, *The Early Universe*,(Addison-Wesley, 1990).
[20] D. H. Lyth and E. D. Stewart, Phys. Lett. B **274**, (1992).
[21] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).
[22] P. Coles and G. F. R. Ellis, *Is the universe open or closed?*, (Cambridge University Press, 1997).
[23] J.D. Barrow and J. Levin, Phys Lett A, **233**, 169, (1997).
[24] P. M. Garnavich, *et al*, to appear in Astrophys. J., astro-ph/9806396.

TABLE I. Invariant sets for the dynamical system given by equations (1.10–1.12).

Label	Set (Ω, x, y)	Dimension	Description
$\mathcal{A}$	$\Omega > 0$ and $\Omega + x^2 + y^2 > 1$	3	positive curvature, non-vacuum
$\mathcal{B}$	$\Omega > 0$ and $\Omega + x^2 + y^2 = 1$	2	zero curvature, non-vacuum
$\mathcal{C}$	$\Omega > 0$ and $\Omega + x^2 + y^2 < 1$	3	negative curvature, non-vacuum
$\mathcal{D}$	$\Omega = 0$ and $\Omega + x^2 + y^2 > 1$	2	positive curvature, no fluid matter
$\mathcal{E}$	$\Omega = 0$ and $\Omega + x^2 + y^2 = 1$	1	zero curvature, no fluid matter
$\mathcal{F}$	$\Omega = 0$ and $\Omega + x^2 + y^2 < 1$	2	negative curvature, no fluid matter

TABLE II. Equilibrium points of the dynamical system (1.10–1.12) and values of the parameters γ and κ for which the equilibrium point is a future attractor.

Label	Ω	x	y	${}^3R$	Eigenvalues	Stability Conditions
P1	0	0	0	-1	$2-3\gamma$ -2 1	Never
P2	0	1	0	0	$6-3\gamma$ 4 $3-\frac{\sqrt{6}}{2}\kappa$	Never
P3	0	-1	0	0	$6-3\gamma$ 4 $3+\frac{\sqrt{6}}{2}\kappa$	Never
P4	0	$\frac{\kappa}{\sqrt{6}}$	$\sqrt{1-\frac{\kappa^2}{6}}$	0	$\kappa^2-3\gamma$ κ^2-2 $\frac{1}{2}\left(\kappa^2-6\right)$	$\gamma<\frac{2}{3},\ \ \kappa^2<3\gamma$ and $\gamma>\frac{2}{3},\ \ \kappa^2<2$
P5	0	$\frac{\sqrt{2}}{\sqrt{3}\kappa}$	$\frac{2}{\sqrt{3}\kappa}$	$\frac{2-\kappa^2}{2}$	$2-3\gamma$ $-1\pm\sqrt{\frac{8}{\kappa^2}-3}$	$\gamma>\frac{2}{3},\ \ \kappa^2>2$
P6	1	0	0	0	$3\gamma-2$ $\frac{1}{2}\left(3\gamma-6\right)$ $\frac{3}{2}\gamma$	Never
P7	$\frac{\kappa^2-3\gamma}{\kappa^2}$	$\frac{\gamma\sqrt{6}}{2\kappa}$	$\frac{\sqrt{12\gamma-6\gamma^2}}{2\kappa}$	0	$3\gamma-2$ $\frac{3}{4}\left[(\gamma-2)\pm\sqrt{(\gamma-2)^2+\frac{8\gamma}{\kappa^2}(\gamma-2)\left(\kappa^2-3\gamma\right)}\right]$	$\gamma<\frac{2}{3},\ \ \kappa^2>3\gamma$